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ORIGINAL RESEARCH

Existence of positive solutions for a fourth-order three-point boundary value problem

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Abstract In this paper, we are concerned with a fourth-order three point boundary value problem. We prove the existence, uniqueness and positivity of solutions by using Leray–Schauder nonlinear alternative, Banach contraction theorem and Guo–Krasnosel'skii fixed point theorem.

Keywords Guo–Kranosel'skii fixed point theorem · Three point boundary value problem · Positive solution · Leray–Schauder nonlinear alternative · Contraction principle

Mathematics Subject Classification 34B10 · 34B15 · 34B18

1 Introduction

The study of the existence of positive solutions for two-point, three-point and multi-point boundary value problems for nonlinear ordinary differential equations has seen a great importance in the recent years.

In fact, it has become an important area of investigation which has received a lot of attentions due to the fact that it has yielded. For more information, one can see the following references [1,4,6,7,9–11,17,19–21].

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The main concern of such a study is the description of the deformations of an elastic beam by means of a fourth-order two-point boundary value problem, where the boundary conditions are given according to the controls at the ends of the beam. One can see here that many papers have been devoted to that problem; namely, the fourth-order two-point boundary value problems. One can refer to the following studies [3,13–15,17,18,22] for more precision.

In 2003, Ma [16]; for instance, described the deformations of an elastic beam whose one end was fixed where as the other one was free considering the fourth-order right focal two-point boundary value problem as follows

$$\begin{aligned} u'''' &= \lambda f\left(t, u\left(t\right), u'\left(t\right)\right), \quad t \in (0,1) \\ u\left(0\right) &= u'\left(0\right) = u''\left(1\right) = u'''\left(1\right). \end{aligned}$$

Moreover, in 2011, Le and Phan [12] showed sufficient conditions for the existence of positive solutions to a multi-point boundary value problem such as

$$\begin{aligned} u'''' &= \lambda f\left(t, x\left(t\right)\right), \quad 0 < t < 1 \\ x^{2k+1}\left(0\right) = 0, \quad x^{2k}\left(1\right) &= \sum_{i=1}^{m-2} \alpha_{ki} x^{2k}\left(\eta_{k_i}\right), \quad k = 0, 1 \end{aligned}$$

by making use of the Guo–Krasnoselskii's fixed point theorem as well as the monotone iterative technique. Where $\lambda > 0$, $0 < \eta_{k1} < \eta_{k2} \ldots < \eta_{km-2} < 1$ $(k = 0, 1)$ and where $1 < \alpha_i < \frac{1}{\eta_i}$, $i = 0, 1$.

In 2006, Zhang, Lishan and Congxin [23], and in 2008, Zhang, Lishan [24]; have studied the existence and uniqueness of nontrivial solutions for a third-order eigenvalue problem generated by the differential equation

$$u'''' = \lambda f\left(t, u\left(t\right), u'\left(t\right)\right), \quad 0 < t < 1,$$

with respectively, the boundary conditions:

$$u\left(0\right) = u'\left(\eta\right) = u''\left(0\right) = 0,$$

and

$$u\left(0\right) = u'\left(\eta\right) = u''\left(1\right) = 0,$$

where $\lambda > 0$ is a parameter, $\frac{1}{2} < \eta < 1$ is a constant, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Their approach is based on Leray–Schauder nonlinear alternative.

Thus, motivated by such papers, we shall study the existence of positive solutions of the following three-point boundary value problem:

$$u''''\left(t\right) + f\left(t, u\left(t\right), u'\left(t\right), u''\left(t\right)\right) = 0, \quad t \in \left(0, 1\right), \tag{1.1}$$

$$u\left(0\right) = u'\left(0\right) = u''\left(0\right) = 0, \quad u''\left(1\right) = \beta u''\left(\eta\right), \tag{1.2}$$

where $f \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$, $\beta > 0$, $0 < \eta < 1$.

Moreover, we shall also make use of Leray–Schauder nonlinear alternative, The Banach contraction theorem and Guo–Krasnoselskii's fixed point theorem.

The organization of this paper is structured as follows:

Section 2: Presents some preliminaries that are used to prove our results.

Section 3: Discusses the existence and the uniqueness of the solution for the BVP (1.1)–(1.2), by using Leray–Schauder nonlinear alternative as well as Banach contraction theorem.

Section 4: Studies the positivity of the solution applying the Guo–Krasnosel'skii fixed point theorem.

Section 5: Gives some examples to illustrate the results obtained.

The assumptions we make throughout this paper are as follows :

$\beta \in \mathbb{R}_+^*$, $0 < \eta < 1$ and $f \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$.

We consider the Banach space $X = C^2[0,1]$, equipped with the norm $\|u\|_X = \max\left\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\right\}$, where $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$.

2 Preliminary Lemmas

In this section, we present several important preliminary lemmas.

Lemma 1 *Let* $\beta\eta \neq 1$ *and* $y \in L^1[0,1]$. *Then, the problem*

$$u'''' + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u'(0) = u''(0) = 0, \quad u''(1) = \beta u''(\eta), \tag{2.2}$$

has a unique solution that is

$$u(t) = \int_0^1 G(t,s)\, y(s)\, ds + \frac{\beta t^3}{6(1-\beta\eta)} \int_0^1 G^*(\eta,s)\, y(s)\, ds, \tag{2.3}$$

where

$$G(t,s) = \frac{1}{6}\begin{cases} (1-s)t^3 - (t-s)^3, & 0 \le s \le t \le 1, \\ (1-s)t^3, & 0 \le t \le s \le 1, \end{cases} \tag{2.4}$$

$$\frac{\partial G(t,s)}{\partial t} = \frac{1}{2}\begin{cases} (1-s)t^2 - (t-s)^2, & 0 \le s \le t \le 1, \\ (1-s)t^2, & 0 \le t \le s \le 1, \end{cases} \tag{2.5}$$

$$G^*(t,s) = \frac{\partial^2 G(t,s)}{\partial t^2} = \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ (1-s)t, & 0 \le t \le s \le 1. \end{cases} \tag{2.5'}$$

Proof By integrating the equation (2.1) over the interval $[0,t]$ for $t \in [0,1]$, we get

$$u(t) = -\frac{1}{6}\int_0^t (t-s)^3\, y(s)\, ds + \frac{1}{6}C_1 t^3 + \frac{1}{2}C_2 t^2 + C_3 t + C_4. \tag{2.6}$$

By the boundary conditions: $u(0) = u'(0) = u''(0) = 0$, we get: $C_4 = C_3 = C_2 = 0$.

And from $u''(1) = \beta u''(\eta)$, we deduce

$$C_1 = \frac{1}{1-\beta\eta}\int_0^1 (1-s)\,y(s)\,ds - \frac{\beta}{1-\beta\eta}\int_0^\eta (\eta - s)\,y(s)\,ds.$$

Substituting C_1, C_2, C_3 and C_4 in (2.6), we obtain

$$\begin{aligned} u(t) &= \frac{1}{6}\left(-\int_0^t (t-s)^3 y(s)\,ds + t^3\int_0^t (1-s)\,y(s)\,ds\right) \\ &\quad + \frac{t^3}{6}\int_t^1 (1-s)\,y(s)\,ds \\ &\quad + \frac{\beta t^3}{6(1-\beta\eta)}\left(\eta\int_0^\eta (1-s)\,y(s)\,ds + \eta\int_\eta^1 (1-s)\,y(s)\,ds\right) \\ &\quad - \frac{\beta t^3}{6(1-\beta\eta)}\int_0^\eta (\eta - s)\,y(s)\,ds. \end{aligned}$$

Thus, by means of elementary operations we get

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\,y(s)\,ds \\ &\quad + \frac{\beta t^3}{6(1-\beta\eta)}\int_0^1 G^*(\eta,s)\,y(s)\,ds. \end{aligned}$$

Which implies Lemma 1. □

We conclude, that in order to discuss the existence of positive solutions, we need some properties of function $G(t,s)$.

Lemma 2 *The function G and its first and second derivatives are nonnegative and for* $t, s \in [0,1]$, *we have*

$$\begin{aligned} &(i)\quad 0 \le G(t,s) \le 2G_1(s), \\ &(ii)\quad 0 \le \frac{\partial}{\partial t}G(t,s) \le G_1(s), \\ &(iii)\quad 0 \le G^*(t,s) \le 2G_1(s), \end{aligned}$$

where $G_1(s) = \frac{1}{2}(1-s)s$.

Proof (i) Let $t, s \in [0,1]$. If $s \le t$, it follows from (2.4) that

$$\begin{aligned} G(t,s) &= \frac{1}{6}\left[(1-s)\,t^3 - (t-s)^3\right] \\ &\le \frac{1}{6}s\left[t^2(1-s) + 3t(t-s)\right] \\ &\le s(1-s) = 2G_1(s). \end{aligned}$$

If $t \leq s$, then

$$G(t,s) = \frac{1}{6}(1-s)t^3 \leq \frac{1}{6}(1-s)s^3 \leq (1-s)s = 2G_1(s).$$

Therefore,

$$0 \leq G(t,s) \leq 2G_1(s), \quad \forall (t,s) \in [0,1] \times [0,1].$$

(ii) Let $t, s \in [0,1]$. If $s \leq t$, it follows from (2.5) that

$$\frac{\partial}{\partial t}G(t,s) = \frac{1}{2}\left[(1-s)t^2 - (t-s)^2\right] = \frac{1}{2}\left[1-s-(1-t)^2\right]s \geq 0,$$
$$\frac{\partial}{\partial t}G(t,s) \leq \frac{1}{2}(1-s)s = G_1(s).$$

If $t \leq s$, it yields

$$\frac{\partial}{\partial t}G(t,s) = \frac{1}{2}t^2(1-s) \leq \frac{1}{2}(1-s)s = G_1(s).$$

Consequently,

$$0 \leq \frac{\partial}{\partial t}G(t,s) \leq G_1(s), \quad \forall (t,s) \in [0,1] \times [0,1].$$

(iii) If $t \leq s$, then $G^*(t,s) = (1-s)t \leq (1-s)s = 2G_1(s)$.
In the case $s \leq t$, we get $G^*(t,s) = (1-t)s \leq (1-s)s = 2G_1(s)$.
This completes the proof of Lemma 2. □

Define an operator $T : X \longrightarrow X$ by

$$Tu(t) = \int_0^1 G(t,s) f\left(s, u(s), u'(s), u''(s)\right) ds + \frac{\beta t^3}{6(1-\beta\eta)} \int_0^1 G^*(\eta,s) f\left(s, u(s), u'(s), u''(s)\right) ds. \qquad (2.7)$$

The function $u \in X$ is a solution of the BVP (1.1)–(1.2) if and only if $Tu(t) = u(t)$.

3 Existence Results

Now we give some existence results for the BVP (1.1)–(1.2).

Theorem 3 *Assume that $\beta\eta \neq 1$ and there exists nonnegative functions $k, h, l \in L^1([0,1], \mathbb{R}_+)$, such that:*

$$|f(t,x,y,z) - f(t,u,v,w)| \leq k(t)|x-u| + h(t)|y-v| + l(t)|z-w|,$$
$$\forall x, y, z, u, v, w \in \mathbb{R}, \quad t \in [0,1], \qquad (3.1)$$

and

$$C = 2\left(1 + \frac{\beta}{|1-\beta\eta|}\right)\int_0^1 G_1(s)\,(k(s) + h(s) + l(s))\,ds < 1, \tag{3.2}$$

then, the BVP (1.1)–(1.2) has a unique solution in X.

Proof We shall prove that T is a contraction. Let $u, v \in X$, then

$$\begin{aligned}|Tu(t) - Tv(t)| &\leq 2\int_0^1 G_1(s)\left|f\left(s, u(s), u'(s), u''(s)\right)\right.\\ &\left.-f\left(s, v(s), v'(s), v''(s)\right)\right|ds\\ &+\frac{\beta}{3|1-\beta\eta|}\int_0^1 G_1(s)\left|f\left(s, u(s), u'(s), u''(s)\right)\right.\\ &\left.-f\left(s, v(s), v'(s), v''(s)\right)\right|ds.\end{aligned}$$

By (3.1), we can reach the following result:

$$\begin{aligned}&|Tu(t) - Tv(t)| \leq \left(2 + \frac{\beta}{3|1-\beta\eta|}\right)\times\\ &\int_0^1 G_1(s)\left[k(s)|u(s) - v(s)| + h(s)\left|u'(s) - v'(s)\right| + l(s)\left|u''(s) - v''(s)\right|\right]ds,\end{aligned}$$

then

$$|Tu(t) - Tv(t)| \leq \|u - v\|_X\left(2 + \frac{\beta}{|1-\beta\eta|}\right)\int_0^1 G_1(s)\,(k(s) + h(s) + l(s))\,ds.$$

Similarly, we get

$$\left|T'u(t) - T'v(t)\right| \leq \|u - v\|_X\left(1 + \frac{\beta}{|1-\beta\eta|}\right)\int_0^1 G_1(s)\,(k(s) + h(s) + l(s))\,ds,$$

and

$$\left|T''u(t) - T''v(t)\right| \leq 2\|u - v\|_X\left(1 + \frac{\beta}{|1-\beta\eta|}\right)\int_0^1 G_1(s)\,(k(s) + h(s) + l(s))\,ds,$$

thanks to (3.2), we get

$$\|Tu - Tv\|_X \leq C\,\|u - v\|_X\,,$$

then, T is a contraction, so it has a unique fixed point which is the unique solution of BVP (1.1)–(1.2). □

Theorem 4 *Assume that* $\beta\eta \neq 1$, $f(t, 0, 0, 0) \neq 0$ *and there exists some nonnegative functions* $k, l, h, m \in L^1[0, 1]$ *such that*

$$|f(t, u, v, w)| \leq k(t)|u| + h(t)|v| + l(t)|w| + m(t), \forall u, v, w \in \mathbb{R}, \quad t \in [0, 1], \tag{3.3}$$

$$2\left(1 + \frac{\beta}{|1 - \beta\eta|}\right) \int_0^1 G_1(s)(k(s) + h(s) + l(s))\, ds < 1. \tag{3.4}$$

Then, the BVP (1.1)–(1.2) *has at least one nontrivial solution* $u^* \in X$.

We need the following Lemma:

Lemma 5 *(Leray-Schauder nonlinear alternative [5]) Let* F *be a Banach space and* Ω *a bounded open subset of* F, $0 \in \Omega$. $T : \overline{\Omega} \to F$ *is a completely continuous operator. Then, either it exists* $x \in \partial\Omega$, $\lambda > 1$ *such that* $T(x) = \lambda x$, *or it exists a fixed point* $x^* \in \overline{\Omega}$.

Proof of Theorem 4. Setting

$$F = 2\left(1 + \frac{\beta}{|1 - \beta\eta|}\right) \int_0^1 G_1(s)(k(s) + h(s) + l(s))\, ds,$$
$$G = 2\left(1 + \frac{\beta}{|1 - \beta\eta|}\right) \int_0^1 G_1(s)\, m(s)\, ds.$$

However, to reach these results, we need to show that T is a completely continuous operator:

1) T continuous. Let $(u_k)_{k\in\mathbb{N}}$ be a convergent sequence to u in X. By applying the upper bounds of the function G and of its first and second derivatives from Lemma 2, we get

$$|Tu_k(t) - Tu(t)| \leq \left(2 + \frac{\beta}{|1 - \beta\eta|}\right) \times$$
$$\int_0^1 G_1(s) \left| f\left(s, u_k(s), u_k'(s), u_k''(s)\right) - f\left(s, u(s), u'(s), u''(s)\right)\right| ds,$$
$$\left|T'u_k(t) - T'u(t)\right| \leq \left(1 + \frac{\beta}{|1 - \beta\eta|}\right) \times$$
$$\int_0^1 G_1(s) \left| f\left(s, u_k(s), u_k'(s), u_k''(s)\right) - f\left(s, u(s), u'(s), u''(s)\right)\right| ds,$$

and

$$\left|T''u_k(t) - T''u(t)\right| \leq 2\left(1 + \frac{\beta}{|1 - \beta\eta|}\right) \times$$
$$\int_0^1 G_1(s) \left| f\left(s, u_k(s), u_k'(s), u_k''(s)\right) - f\left(s, u(s), u'(s), u''(s)\right)\right| ds,$$

which imply,

$$\|Tu_k - Tu\|_X \leq 2\left(1 + \frac{\beta}{|1-\beta\eta|}\right) \times$$
$$\int_0^1 G_1(s)\left|f\left(s, u_k(s), u_k'(s), u_k''(s)\right) - f\left(s, u(s), u'(s), u''(s)\right)\right| ds,$$

since $G_1(s) \leq \frac{1}{8}$ then

$$\|Tu_k - Tu\|_X \leq \frac{1}{4}\left(1 + \frac{\beta}{|1-\beta\eta|}\right) \times$$
$$\int_0^1 \left|f\left(s, u_k(s), u_k'(s), u_k''(s)\right) - f\left(s, u(s), u'(s), u''(s)\right)\right| ds,$$

applying Lebesgue's dominated convergence theorem it yields $\|Tu_k - Tu\|_X \to 0$, when $k \to +\infty$. This implies that T is continuous.

2) Let $B_r = \{u \in X : \|u\|_X \leq r\}$ a bounded subset. We shall prove that $T(B_r)$ is relatively compact:

(i) $T(B_r)$ uniformly bounded. For some $u \in B_r$, using (3.3) we obtain

$$|Tu(t)| \leq \left(2 + \frac{\beta}{|1-\beta\eta|}\right)\|u\|_X \int_0^1 G_1(s)(k(s) + h(s) + l(s))\, ds$$
$$+ \left(2 + \frac{\beta}{|1-\beta\eta|}\right)\int_0^1 G_1(s)\, m(s)\, ds.$$

Similarly, we have

$$\left|T'u(t)\right| \leq \left(1 + \frac{\beta}{|1-\beta\eta|}\right)\|u\|_X \int_0^1 G_1(s)(k(s) + h(s) + l(s))\, ds$$
$$+ \left(1 + \frac{\beta}{|1-\beta\eta|}\right)\int_0^1 G_1(s)\, m(s)\, ds.$$

And,

$$\left|T''u(t)\right| \leq 2\left(1 + \frac{\beta}{|1-\beta\eta|}\right)\|u\|_X \int_0^1 G_1(s)(k(s) + h(s) + l(s))\, ds$$
$$+2\left(1 + \frac{\beta}{|1-\beta\eta|}\right)\int_0^1 G_1(s)\, m(s)\, ds.$$

From the above inequalities we have $\|Tu\|_X \leq F\|u\|_X + G \leq Fr + G$. Thus, $T(B_r)$ is uniformly bounded.

(ii) $T(B_r)$ equicontinuous. Let $t_1, t_2 \in [0,1]$, $u \in B_r$,

$$L = \max\left\{\left|f\left(s, u(s), u'(s), u''(s)\right)\right|, s \in [0,1], \|u\|_X \le r\right\},$$

therefore, we have:

$$\begin{aligned}|Tu(t_1) - Tu(t_2)| &\le L\,|t_2 - t_1| \\ &\left[\left|t_1\left(3 + \frac{t_1}{2}\right)(t_1 + t_2) + \frac{t_1}{2}\left(t_1^2 + t_2^2 + t_1 t_2\right)\right|\right. \\ &\left.+\frac{\left(t_1^2 + t_2^2 + t_1 t_2\right)\beta}{6\,|1 - \beta\eta|}\int_0^1 \left|G^*(\eta, s)\right| ds\right].\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left|T'u(t_1) - T'u(t_2)\right| &\le L\,|t_2 - t_1| \\ &\left[1 - t_2^2 + t_1(t_1 - t_2 + 3) + \frac{(t_1 + t_2)\beta}{2\,|1 - \beta\eta|}\int_0^1 \left|G^*(\eta, s)\right| ds\right].\end{aligned}$$

We also have:

$$\begin{aligned}\left|T''u(t_1) - T''u(t_2)\right| &\le L\,|t_2 - t_1| \\ &\left[1 + (t_1 - t_2) + \frac{1}{2}(3t_2 - 5t_1) + \frac{\beta}{|1 - \beta\eta|}\int_0^1 \left|G^*(\eta, s)\right| ds\right].\end{aligned}$$

These show that, $|Tu(t_1) - Tu(t_2)| \underset{t_1 \to t_2}{\longrightarrow} 0$, $\left|T'u(t_1) - T'u(t_2)\right| \underset{t_1 \to t_2}{\longrightarrow} 0$ and $\left|T''u(t_1) - T''u(t_2)\right| \underset{t_1 \to t_2}{\longrightarrow} 0$. Consequently, $T(B_r)$ is equicontinuous. From Arzela-Ascoli theorem, we deduce that T is a completely continuous operator.

□

Proof Now, from the continuity of f and the fact that $f(t, 0, 0, 0) \ne 0$, we conclude that there exists an interval $[\sigma_1, \sigma_2] \subset [0, 1]$ such that $\min_{\sigma_1 \le t \le \sigma_1} |f(t, 0, 0, 0)| > 0$ and then $G > 0$ since $m(t) \ge |f(t, 0, 0, 0)| > 0$ on $[\sigma_1, \sigma_2]$. Let $M = G(1 - F)^{-1}$, $\Omega = \{u \in X : \|u\| < M\}$ and $u \in \partial\Omega$, $\lambda > 1$ such that $Tu = \lambda u$, then with the help of (3.3) it yields

$$\begin{aligned}|Tu(t)| &\le \|u\|_X \left(2 + \frac{\beta}{|1 - \beta\eta|}\right)\int_0^1 G_1(s)\left(k(s) + h(s) + l(s)\right) ds \\ &+ \left(2 + \frac{\beta}{|1 - \beta\eta|}\right)\int_0^1 G_1(s)\, m(s)\, ds.\end{aligned}$$

Moreover,

$$\left|T'u\left(t\right)\right| \leq \|u\|_X \left(1+\frac{\beta}{|1-\beta\eta|}\right)\int_0^1 G_1\left(s\right)\left|\left(k\left(s\right)+h\left(s\right)+l\left(s\right)\right)\right| ds$$
$$+\left(1+\frac{\beta}{|1-\beta\eta|}\right)\int_0^1 G_1\left(s\right) m\left(s\right) ds.$$

And,

$$\left|T''u\left(t\right)\right| \leq 2\|u\|_X \left(1+\frac{\beta}{|1-\beta\eta|}\right)\int_0^1 G_1\left(s\right)\left|\left(k\left(s\right)+h\left(s\right)+l\left(s\right)\right)\right| ds$$
$$+2\left(1+\frac{\beta}{|1-\beta\eta|}\right)\int_0^1 G_1\left(s\right) m\left(s\right) ds.$$

This shows that

$$\lambda M = \|Tu\|_X \leq F\,||u||_X + G = FM + G.$$

From this we get

$$\lambda \leq F + \frac{G}{M} = F + \frac{G}{G\left(1-F\right)^{-1}} = F + \left(1-F\right) = 1.$$

However, this contradicts $\lambda > 1$. By applying Lemma 5, we deduce that T has a fixed point $u^* \in \overline{\Omega}$ and so, the BVP (1.1)–(1.2) has a nontrivial solution $u^* \in X$. Consequently, the proof is complete.

4 Positive Results

In this section, we shall discuss the existence of positive solutions of BVP (1.1)–(1.2). By making the following additional assumptions:

(*Q1*) $f(t,u,v,w) = a(t) f_1(u,v,w)$ where $a \in C([0,1], \mathbb{R}_+)$ and $f_1 \in C(\mathbb{R}\times\mathbb{R}^2, \mathbb{R}_+)$.

(*Q2*) $\int_0^1 G_1\left(s\right) a(s) ds > 0$.

In fact, we need the lower bounds of functions G and its derivatives.

Lemma 6 *Let* $t \in [\tau_1, \tau_2]$, $s \in [0,1]$, $0 < \tau_1 < \tau_2 < 1$, *then*

$$i)\quad G\left(t,s\right) \geq \frac{1}{3}\tau_1^3 G_1\left(s\right),$$
$$ii)\quad \frac{\partial}{\partial t} G\left(t,s\right) \geq \tau_1^2 G_1\left(s\right),$$
$$iii)\quad G^*\left(t,s\right) \geq \gamma G_1\left(s\right),$$

where $\gamma = \min\left\{2\tau_1^2, 2\left(1-\tau_2\right)\right\}$.

Proof i) If $s \leq t$, then

$$G(t,s) \geq \frac{1}{6}s(1-s)t^3 + \frac{1}{6}t^3(1-s)^3 - \frac{1}{6}(t-s)^3,$$

$$G(t,s) \geq \frac{1}{6}s(1-s)t^3 + \frac{1}{6}s(1-t)\left[t^2(1-s)^2 + t(1-s)(t-s) + (t-s)^3\right],$$

$$G(t,s) \geq \frac{1}{6}s(1-s)t^3 \geq \frac{1}{3}\tau_1^3 G_1(s).$$

If $t \leq s$, it follows that

$$G(t,s) = \frac{1}{6}(1-s)t^3 \geq \frac{1}{6}s(1-s)t^3 \geq \frac{1}{3}\tau_1^3 G_1(s).$$

ii) If $s \leq t$, then

$$\begin{aligned}\frac{\partial}{\partial t}G(t,s) &= \frac{1}{2}\left(2t - t^2 - s\right)s \\ &= \frac{1}{2}st^2(1-s) + \frac{1}{2}(1-t)\left[(t-s) + (1-s)t\right]s.\end{aligned}$$

Thus,

$$\frac{\partial}{\partial t}G(t,s) \geq \frac{1}{2}t^2(1-s)s \geq \tau_1^2 G_1(s).$$

Now in the case $t \leq s$, we get

$$\frac{\partial}{\partial t}G(t,s) = \frac{1}{2}t^2(1-s) \geq \frac{1}{2}t^2(1-s)s \geq \tau_1^2 G_1(s).$$

iii) If $t \leq s$, then

$$G^*(t,s) = (1-s)t = (1-s)s\frac{t}{s} \geq (1-s)s\tau_1 \geq 2\tau_1^2 G_1(s).$$

In the case $s \leq t$, it yields

$$G^*(t,s) = (1-t)s = \frac{1-t}{1-s}(1-s)s \geq$$
$$(1-\tau_2)(1-s)s \geq 2(1-\tau_2)G_1(s),$$

from here we conclude that $G^*(t,s) \geq \gamma G_1(s)$ on $[\tau_1, \tau_2] \times [0,1]$. This completes the proof. □

Lemma 7 *Under the assumptions* (Q1)–(Q2) *and if* $0 < \beta\eta < 1$ *and if* u *is a solution of the BVP* (1.1)–(1.2) *then* u *is nonnegative and satisfies*

$$\min_{t\in[\tau_1,\tau_2]}\left(u(t) + u'(t) + u''(t)\right) \geq \lambda\left[2\left(1 + \frac{\beta}{1-\beta\eta}\right)\right]^{-1}||u||_X,$$

where $\lambda = \min\left\{\gamma, \tau_1^2, \frac{1}{3}\tau_1^3\right\}$.

Proof Let u be a solution of the BVP (1.1)–(1.2), then

$$u(t) = Tu(t) = \int_0^1 G(t,s)\, a(s)\, f_1\left(u(s), u'(s), u''(s)\right) ds + \frac{\beta t^3}{6(1-\beta\eta)} \int_0^1 G^*(\eta,s)\, a(s)\, f_1\left(u(s), u'(s), u''(s)\right) ds.$$

From the assumptions (Q1)–(Q2) and the positivity of G and G^*, it is obvious that u is nonnegative. Now using Lemma 2 we get

$$\|u\|_\infty \le \left(2 + \frac{\beta}{1-\beta\eta}\right) \int_0^1 G_1(s)\, a(s)\, f_1\left(u(s), u'(s), u''(s)\right) ds.$$

On the other hand, for any $t \in [\tau_1, \tau_2]$ by Lemma 6, we get

$$u(t) \ge \frac{1}{3}\tau_1^3 \left(2 + \frac{\beta}{1-\beta\eta}\right)^{-1} \|u\|_\infty.$$

Therefore, we have:

$$\min_{t\in[\tau_1,\tau_2]} u(t) \ge \frac{1}{3}\tau_1^3 \left(2 + \frac{\beta}{1-\beta\eta}\right)^{-1} \|u\|_\infty.$$

Similarly, for any $t \in [\tau_1, \tau_2]$, we get

$$\min_{t\in[\tau_1,\tau_2]} u'(t) \ge \tau_1^2 \left(1 + \frac{\beta}{1-\beta\eta}\right)^{-1} ||u'||_\infty.$$

And

$$\min_{t\in[\tau_1,\tau_2]} u''(t) \ge \frac{\gamma}{2} \left(1 + \frac{\beta}{1-\beta\eta}\right)^{-1} ||u''||_\infty.$$

Finally we get:

$$\min_{t\in[\tau_1,\tau_2]} \left(u(t) + u'(t) + u''(t)\right) \ge \lambda \left[2\left(1 + \frac{\beta}{1-\beta\eta}\right)\right]^{-1} ||u||_X.$$

This finishes the proof. □

Define the set K by

$$K = \Big\{ u \in X,\ u(t) \geq 0, \quad t \in [0,1], \quad \min_{t\in[\tau_1,\tau_2]} \left(u(t) + u'(t) + u''(t)\right) \\ \geq \lambda \left[2\left(1 + \frac{\beta}{1-\beta\eta}\right)\right]^{-1} ||u||_X \Big\}.$$

We prove easily that K is a non-empty closed and convex subset of X, then it is a cone.

Lemma 8 *The operator T is completely continuous and satisfies $T(K) \subseteq K$.*

Proof To prove that it suffices to apply Arzela-Ascoli theorem, and following the proof of Lemma 7, one can show that $T(K) \subset K$. □

To establish the existence of positive solutions of BVP (1.1)–(1.2), we shall employ the following Guo–Krasnosel'skii fixed point theorem [8].

Theorem 9 *Let E be a Banach space and let $K \subset E$ be a cone. We assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let $\mathcal{A} : K \cap \left(\overline{\Omega_2}\backslash\Omega_1\right) \to K$ be a completely continuous operator. In addition, we suppose either*

(i) $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_1$, *and* $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_2$ *or*
(ii) $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_1$, *and* $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_2$

holds. Then $\mathcal{A}$ has a fixed point in $K \cap \left(\overline{\Omega_2}\backslash\Omega_1\right)$.

The main result of this section is the following:

Theorem 10 *Under the assumptions (Q_1) and (Q_2) and if $0 < \beta\eta < 1$, $f_0 = 0$ and $f_\infty = \infty$. Then, the BVP (1.1)–(1.2) has at least one nontrivial positive solution. Where*

$$f_0 = \lim_{(|u|+|v|+|w|)\to 0} \frac{f_1(u,v,w)}{|u|+|v|+|w|},$$
$$f_\infty = \lim_{(|u|+|v|+|w|)\to \infty} \frac{f_1(u,v,w)}{|u|+|v|+|w|}.$$

Proof We shall prove that the problem BVP (1.1)–(1.2) has at least one positive solution. For this we use Theorem 9. From Lemma 8 we know that T is completely continuous and that $T\left(K \cap \left(\overline{\Omega_2}\backslash\Omega_1\right)\right) \subset K$. Now, since $f_0 = 0$, then for any $\varepsilon > 0$, there exists $\delta_1 > 0$, such that $f_1(u,v,w) \leq \varepsilon(|u| + |v| + |w|)$, for $|u|+|v|+|w| < \delta_1$. Let Ω_1 be an open set in X defined by $\Omega_1 = \left\{y \in X / ||y||_X < \delta_1\right\}$. Then, for any $u \in K \cap \partial\Omega_1$, it yields:

$$\|Tu\|_\infty \leq \varepsilon \|u\|_X \left(2 + \frac{\beta}{1-\beta\eta}\right) \int_0^1 G_1(s)\, a(s)\, ds,$$

and

$$\left\|T'u\right\|_{\infty} \leq \varepsilon \left\|u\right\|_X \left(1+\frac{\beta}{1-\beta\eta}\right)\int_0^1 G_1(s)\,a(s)\,ds.$$

And

$$\left\|T''u\right\|_{\infty} \leq 2\varepsilon \left\|u\right\|_X \left(1+\frac{\beta}{1-\beta\eta}\right)\int_0^1 G_1(s)\,a(s)\,ds.$$

If we choose $\varepsilon = \left[2\left(1+\frac{\beta}{1-\beta\eta}\right)\int_0^1 G_1(s)\,a(s)\,ds\right]^{-1}$, then it yields

$$\|Tu\|_X \leq \|u\|_X, \quad \forall u \in K \cap \partial\Omega_1.$$

Now, from $f_\infty = \infty$, we conclude that for any $M > 0$, there exists $H > 0$, such that $f_1(u, v, w) \geq M(|u| + |v| + |w|)$ for $|u| + |v| + |w| \geq H$. Let

$$H_1 = \max\left\{2\delta_1, \frac{2H}{\lambda}\left(1+\frac{\beta}{1-\beta\eta}\right)\right\},$$

and denote by Ω_2 the open set $\Omega_2 = \left\{y \in X / \|y\|_X < H_1\right\}$, then $\overline{\Omega_1} \subset \Omega_2$. Let $u \in K \cap \partial\Omega_2$; then,

$$\begin{aligned}\min_{t\in[\tau_1,\tau_2]}\left\{u(t)+u'(t)+u''(t)\right\} &\geq \lambda\left[2\left(1+\frac{\beta}{1-\beta\eta}\right)\right]^{-1}\|u\|_X\\ &= \lambda\left[2\left(1+\frac{\beta}{1-\beta\eta}\right)\right]^{-1}H_1 \geq H.\end{aligned}$$

Now let $t \in [\tau_1, \tau_2]$, taking into account Lemma 6, we obtain

$$Tu(t) \geq M\frac{\lambda^2}{2}\left(1+\frac{\beta}{1-\beta\eta}\right)^{-1}\left(1+\frac{\gamma\beta}{2(1-\beta\eta)}\right)\|u\|_X\int_{\tau_1}^{\tau_2} G_1(s)\,a(s)\,ds.$$

Similarly,

$$T'u(t) \geq M\frac{\lambda^2}{2}\left(1+\frac{\beta}{1-\beta\eta}\right)^{-1}\left(1+\frac{\gamma\beta}{2(1-\beta\eta)}\right)\|u\|_X\int_{\tau_1}^{\tau_2} G_1(s)\,a(s)\,ds.$$

And

$$T''u(t) \geq M\frac{\lambda^2}{2}\left(1+\frac{\beta}{1-\beta\eta}\right)^{-1}\left(1+\frac{\tau_1\beta}{2(1-\beta\eta)}\right)\|u\|_X\int_{\tau_1}^{\tau_2} G_1(s)\,a(s)\,ds.$$

By choosing

$$M = \left[\frac{\lambda^2}{2} \left(1 + \frac{\beta}{1 - \beta\eta} \right)^{-1} \left(1 + \frac{\zeta\beta}{2(1 - \beta\eta)} \right) \int_{\tau_1}^{\tau_2} G_1(s)\, a(s)\, ds \right]^{-1},$$

where $\zeta = \min(\gamma, \tau_1)$ we get:

$$\|Tu\|_X \geq \|u\|_X, \forall u \in K \cap \partial\Omega_2$$

By the first part of Theorem 9, T has at least one fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$, which is a nontrivial positive solution of BVP (1.1)–(1.2). This achieves the proof of Theorem 10. □

5 Examples

In order to illustrate our results, we give these examples:

Example 11 We consider the following boundary value problem

$$\begin{cases} u'''' + tu + t^2 u' + t^3 u'' = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, & u''(1) = \beta u''(\eta). \end{cases} \tag{J1}$$

We set $\beta = \frac{1}{3}$, $\eta = \frac{1}{4}$, and $f(t, u, v, w) = tu + t^2 v + t^3 w$. We can choose

$$k(t) = t,\ h(t) = t^2,\ l(t) = t^3, \quad t \in [0, 1]$$

$k, h, l \in L^1[0, 1]$ are nonnegative functions, and

$$\begin{aligned} |f(t, x, y, z) - f(t, u, v, z)| &\leq t|x - u| + t^2|y - v| + t^3|z - w| \\ &\leq k(t)|x - u| + h(t)|y - v| + l(t)|z - w| \end{aligned}$$

with,

$$C = 2\left(1 + \frac{\beta}{|1 - \beta\eta|}\right) \int_0^1 G_1(s)(k(s) + h(s) + l(s))\, ds = 0.227 < 1,$$

hence, by Theorem 3, the boundary value problem (J1) has a unique solution in X.

Example 12 We consider the following boundary value problem

$$\begin{cases} u'''' + 2tu + t^2 u' \sin t + \frac{t^3}{3} u'' + \sin 2t = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, & u''(1) = \beta u''(\eta). \end{cases} \tag{J2}$$

We set $\beta = \frac{1}{2}, \quad \eta = \frac{1}{4}$. Now if we estimate f as

$$\begin{aligned} |f(t,u,v,w)| &\le 2t\,|u| + t^2\,|v| + \frac{t^3}{3}\,|w| + \sin 2t \\ &\le k(t)\,|u| + h(t)\,|v| + l(t)\,|w| + m(t), \end{aligned}$$

then, we can choose $k(t) = 2t, \quad l(t) = t^2, \quad h(t) = \frac{t^3}{3}, \quad m(t) = \sin 2t, \quad t \in [0,1]$ with

$$2\left(1 + \frac{\beta}{|1 - \beta\eta|}\right)\int_0^1 G_1(s)\,(k(s) + h(s) + l(s))\,ds = 0.357\,94 < 1,$$

k, l, h and $m \in L^1[0,1]$ which are nonnegative functions. Hence, by Theorem 4, the boundary value problem (J2) has at least one nontrivial solution, $u^* \in X$.

Example 13 We consider the following boundary value problem

$$\begin{cases} u'''' + t^2u^2 + \frac{t^2}{4}\left(u'\right)^2 + \frac{t^2}{9}\left(u''\right)^2 = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \quad u''(1) = \beta u''(\eta). \end{cases} \tag{J3}$$

where $0 < \beta\eta < 1$. Knowing that:

$$f(t,u,v,w) = t^2\left(u^2 + \frac{1}{4}v^2 + \frac{1}{9}w^2\right) = a(t)\,f_1(u,v,w),$$

$$a(t) = t^2 \in C([0,1], \mathbb{R}_+), \; f_1(u,v,w) \in C\left(\mathbb{R}_+ \times \mathbb{R}^2, \mathbb{R}_+\right), \text{ we get}$$

$$\lim \frac{f_1(u,v,w)}{|u| + |v| + |w|} = \begin{cases} 0, \text{ if } (|u| + |v| + |w|) \to 0, \\ \infty, \text{ if } (|u| + |v| + |w|) \to \infty. \end{cases}$$

So, we have the superlinear case $f_0 = 0$ and $f_\infty = \infty$; consequently, by Theorem 10, the BVP (J3) has at least one positive solution.

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